SOME NOTES ON SHANNON'S LIMIT:
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The most widely misunderstood and misused equation in digital communications is the channel capacity equation. (Shannon). This equation is based on earlier work by Nyquist. (Sampling Theorem). The general mistake is in assuming the "Normalized Bandwidth" is always used.

All digital communication takes place in the form of pulses, usually square wave pulses. Each pulse has a time duration 'τ' and an associated repetition rate. If the pulses are phase reversing end to end, as in Coded BPSK or VMSK, then the repetition Rate is $1/\tau$ symbols per sec. For Coded BPSK and VMSK, this is $1/\tau$ bits per second, since there is only one bit per symbol.

Normally, the signals are sampled at the minimum Nyquist sampling rate, which is equal to two samples per bit at the baseband frequency f_m, which is ½ of the bit rate f_b. Thus the minimum sampling rate 'W' is equal to the bit rate, = $1/\tau$.

W is defined as the number of samples per second (Schwartz- [4] pp324 and equation 6-134), and as the number of transmissions per second (which equals the repetition rate, or bit rate) by Proakis and Saleh [5] (pp736).

IT IS NOT THE BANDWIDTH USED, IT IS THE NYQUIST SAMPLING RATE. (This is where many engineers go astray). It is also the "normalized bandwidth".

"The system channel capacity 'C' is obtained by multiplying the number of samples per second by the information per sample." (Schwartz, [4] pp 324 and equation 6-134).

The channel capacity is defined by Schwartz (Eqs. 6-125 and 2-149) as:

$$C = (1/\tau) \log_2 n$$

A) The classic form of Shannon's equation is expressed as:

(1) $f_b = W \log_2 \{ 1 + C/N \} \qquad n = \{ 1 + C/N \} \qquad W = (1/\tau)$

Where C/N = (Bit Rate/Filter Bandwidth)(E_b / N_0) and f_b is the channel capacity, which is equal to the bit rate when the limit is reached.

B) Taub and Shilling express the relationship with substituted terms as shown in [2]

$$\frac{f_b}{W} = \log_2\left(1 + \frac{f_b}{B}\frac{E_b}{\eta}\right) \quad [2]$$

Use the term f_b for both channel capacity and bit rate, and B for the actual filter noise bandwidth (not the normalized bandwidth = W). f_b is substituted for C to avoid a conflict of terms in the carrier to noise ratio C/N.

Every term in this equation but W is variable. Something must remain fixed. W is the minimum sampling rate = $1/\tau$ symbols per sec., but W can also be <u>the Nyquist bandwidth,</u> or normalized bandwidth, defined as the minimum bandwidth that can be used to pass the signal. The Nyquist Sampling theorem is derived from that bandwidth. Basically the theorem says "you must sample once every bit to see if it is there or not".

Also:

f_b / W = Bits / Symbol

f_b / B = Bandwidth Efficiency

(f_b / B)(E_b / N_0) = C / N

These equalities are accepted throughout the world as fact. Note also that for most modulation methods *:

bits/symbol = bandwidth efficiency = bit rate/(noise bandwidth)

and: bits/symbol = bit rate/Nyquist bandwidth = bit rate / sampling rate

***These equalities are true <u>only if the bandwidth of the filter equals the Nyquist bandwidth (Normalized Bandwidth)</u>. If not, the *bandwidth efficiency* <u>does not = *bits/symbol.*</u>**

If W is not held fixed at the Nyquist bandwidth, which results in f_b / W not being = Bits / Symbol, <u>the equation gives false results.</u>

C) Consider the case of 1024 QAM modulation as an example. (10 bits per symbol).

For B = W and (f_b/W) = 10, then, as shown in (3):

(3) $10 = \log_2 \{ 1+ 10\, E_b / N_0 \}$

When E_b / N_0 = 100 (or 20dB), the equation balances to result in Shannon's Limit. This is the expected and widely published result.

D) Still considering 1024 QAM, now double the receiver noise bandwidth (cutting the bandwidth efficiency in half) and improperly cutting the value of W in half so that (f_b/W) and (f_b / B) = 5, then, in (4):

(4) $5 = \log_2 \{ 1+ 5\, E_b / N_0 \}$

Solving through, Shannon's Limit would then be 7.9 dB. This implies that broadening the filter's noise bandwidth improves the transmission system. This is obviously incorrect. If it were correct, engineers wouldn't try so hard to obtain narrow band filters, they would merely use broader and broader filters.

The mistake is in changing W so that 5 bits per symbol is implied, or that the data can be sampled once every other bit, which is a violation of the sampling theorem.

E) If, however, f_b/ W = Bits / Symbol and f_b/B = Bandwidth Efficiency, then in (5) :

(5) $10 = \log_2 \{1+ 5 E_b / N_0\}$

Now, E_b / N_0 must = 200, and Shannon's Limit is 23 dB. This is the correct answer. Neither the bits/symbol, nor the (bit rate)/(sample rate) has been changed, but there is twice as much noise bandwidth. W was maintained at the Nyquist bandwidth.

F) Now consider VMSK/2. As in example (4), let B = W so that (f_b/W) and (f_b / B) = 100, then, in (6):

(6) $100 = \log_2 \{1+ 100 E_b / N_0\}$ f_b/B = Bandwidth Efficiency = 100

Shannon's Equation balances only when E_b / N_0 has about 300 dB Carrier to Noise ratio (with Q = 100 b/s/Hz = f_b/B). This is obviously impossible! That much available power does not exist in the U.S. It amounts to a few billion MegaWatts of power.

The mistake is in not equating W to $1/\tau$. The sampling rate cannot be reduced to 1/100 th of the bit rate, and 100 bits per symbol are not being used. VMSK is a one bit per symbol method and must be analyzed as such.

G) Instead, let Bits/Symbol (f_b /W) = 1 and Bandwidth efficiency (f_b/B) = 100. Then B does not equal W (7) $B \neq W$ and $B \neq 1/\tau$.

and, in (8):

:

(8) $1 = \log_2 \{1+ 100 E_b / N_0\}$

To be corrected later--------

Shannon's Limit with this equation is reached when E_b / N_0 is about -20 dB (with Q = 100 b/s/Hz).

But this is also a very controversial answer for most engineers, since it is below the accepted minimum for Shannon's Limit calculated using the normalized bandwidth. This very low value occurs because the bit energy E_b was not reduced to reflect the nature of the VMSK signal. Note the equation:

(9) **SNR = β^2 [bit rate /filter bandwidth] E_b/η** From Feher [3]

SNR = C/N

VMSK in its optimum form is phase modulation, where β is less than 1/4 radian. There is considerable loss of SNR when β is less than one radian. (See Eq. 16)

This equation can be restated as: **SNR = [Modulation loss][Processing gain] E_b/η**

Eq. 8 did not take into consideration the modulation loss. If **β^2 = 1/100, then**

(10) **SNR = [Modulation loss][Processing gain] E_b/η** = [1/100][100] **E_b/η = E_b/η**

This is the same as for BPSK modulation. C/N = **E_b/η**

Processing Gain = Bandwidth Efficiency = **[bit rate /filter bandwidth]**

The differences in these examples are that the *bandwidth efficiency* does not always = *bits/symbol and the bandwidth used is not always the "normalized Bandwidth"*. [6].

***In any modulation method, the bandwidth efficiency can be changed by the filter, but the bits per symbol remains unchanged. Also the Bit Rate/Sample Rate must remain unchanged. Consider also that all modulation methods have loss due to the modulation method that must be overcome by the processing gain = bandwidth efficiency* = (f_b/B).**

H) Apply the above relationships to GMSK and BPSK where a filter bandwidth relationship BT = .33 is used. The filter is only 1/3 as wide as the Nyquist, or normalized, bandwidth.

Incorrectly: $f_b = W \log_2 \{ 1 + C/N \}$ $\quad$ $f_b /.33 = \log_2 \{ 1 +(f_b /.33) E_b / N_0 \} = 3$

The method does not use 3.0 bits per symbol. It remains = 1.0.

$3 = \log_2 \{ 1 +(f_b /.33) E_b / N_0 \} = \log_2 \{ 1 +(3) E_b / N_0\}$

Incorrectly used: E_b / N_0 **) must be raised until** $(1+E_b / N_0) = 8$.

Keeping the bits per symbol at 1.0 where it belongs ($f_b = W$ **):**

(11) $1 = \log_2 \{ 1 +(3) E_b / N_0\}$

According to (11), E_b / N_0 can be reduced by the amount the noise bandwidth has been reduced. Or, the E_b / N_0 value can be maintained and the channel capacity is doubled.

?) Question: If 1024 QAM requires 10 Bits / Symbol to be maintained and all methods demand that W = 1/τ to comply with Shannon's Limit, then, doesn't the same rule apply to VMSK as well? Accordingly, the correct application of Shannon's Limit requires that, as in equation (8), B (bandwidth used) should not equal W = 1/τ (the sampling rate) in all cases. What happens when a Nyquist filter with excess bandwidth α is used as in (5)? Or when the filter is less than the Nyquist bandwidth as in (9)? The key to this analysis is the fact that for VMSK, Bits /

Symbol always has a value of "1", which is not related to the filter bandwidth. The filter bandwidth changes C/N only. (or the bandwidth efficiency). <u>Bits per symbol must conform to reality with any modulation method.</u>

Using single sideband converts the VMSK signal to PM, where only the J_0 and $-J_1$ signals are present. The original bandwidth occupied is equal to the Nyquist baseband bandwidth. Transmitting with suppressed carrier removes the J_0 frequency, leaving only the $-J_1$ to be transmitted, which is a single frequency.

The received and true bandwidth efficiency however, when defined as [bit rate]/[filter bandwidth], is much larger than 1. [bit rate /filter bandwidth] is also defined as processing gain.

Basically, Shannon's Limit says you can't tell the signal from the noise when this limit is reached. This is also a restatement of " the limit is reached when SNR = 1 = 0 dB".

The Signal to Noise ratio of a system is defined by the equation:

(9) $C/N = SNR = \beta^2$ **[bit rate /filter bandwidth]** E_b/η

or

(10) $C/N = SNR =$ **[Bit rate /Filter bandwidth][Modulation Energy Loss]** E_b/η

Or, in this example:) $C/N = SNR =$ **[Modulation loss =** β^2 **][Processing gain]** E_b/η

Note also the following equation for the 2 level bit error rate.

(12) $$BER = P_e = (1/2)\mathrm{erfc}(SNR)^{1/2}$$

Here it is seen that the error probability is 50/50 when <u>'SNR'</u> = 0 dB. Eq. (8) applies when the bandwidth is varied, as in 1024QAM. The formulas (3) and (4) do not apply.

The value of -1.6 dB for Shannon's Limit applies only when the "normalized bandwidth" is used. It applies mostly to OFSK, where a large number of channels are used side by side, spaced orthogonally. It can apply to other methods where the signal Nyquist bandwidth is deliberately spread and the sampling rate is increased proportionally. The modulation in OFDM is on/off keying. The receiver contains a single filter for each of the channels. Since there are 'N' channels in use, they must be all be sampled at the same time. $W = 1/\tau = N f_b$.

The bandwidth efficiency, or bits per symbol, is less than 1.0

Eq. 1 then becomes correctly- for N channels:

(14) $f_b = W \log_2 \{ 1 + C/\underline{N} \}$ **or** $f_b / N f_b = \log_2 \{ 1 +(f_b /B) E_b/\eta \}$

(N is shown as having two meanings N and <u>N</u>).

(15) $Pe = [(M-1)/2]\text{erfc}[NE_b/2n]^{1/2}$ For OFDM .

When solved for very large N, with normalized bandwidth, the limit becomes -1.6 dB.

SNR for OFSK is [N $E_b/2\eta$]. Thus as N is made larger, **E_b can be made smaller and smaller, so that it may actually fall below -1.6 dB. Most certainly SNR in the Pe equation can have a large negative value. The above relationships are from Taub and Schilling [2]. See also Bernard Sklar [6].**

To return to Eq. 10 using VMSK/2

C/N = SNR = [Bit rate /Filter bandwidth][Modulation Energy Loss] E_b/η

[Bit rate /Filter bandwidth] is typically between 250 and 300 = 24 dB.

[Modulation Energy Loss] varies from 1/25 to 0ver 1/100. At 1/100 the loss is 20 dB.

C/N can be = (1/100)(300) E_b/η = 3 E_b/η. Now look at Eq. 11. The C/N for VMSK/2 is seen to be the same as for BPSK with a BT = .33 filter. This value has been measured in an operating system.

Shannon's Limit is calculated as:

(16) $f_b = W \log_2 \{1 + C/\underline{N}\}$ or, $f_b = W \log_2 \{1 + 3 E_b/\eta\}$

$f_b/W = \log_2 \{1 + 3 E_b/\eta\} = 1$. **$E_b/\eta$** can equal 1/3 to make the equation balance, or -4.7 dB lower, <u>which does not calculate to be the generally accepted -1.6 dB for S.L.</u>

There is nothing unusual in using the relationship from Feher [3]:

(9) C/N = SNR = β^2 [bit rate /filter bandwidth] E_b/η

or

(10) C/N = SNR = [Bit rate /Filter bandwidth][Modulation Energy Loss] E_b/η

Narrow band FM fits in this formula.

SNR = C/N = $a\beta^2 E_b/\eta$ (from [7]), where 'a' varies from .5 to 3 depending upon various factors. It is normally 3/2 for wideband FM and 1.0 for narrow band FM. The amount of **E_b/η** that survives when the modulation energy loss β^2 changes is in agreement with this formula. Let the modulation index $\beta = .1$, then only 1/100 of the E_b/η survives. The processing gain [bit rate /filter bandwidth] is fixed at 1.0, since the Bessel products are separated by an amount equal to the bit rate, so the C/N is very poor.

VMSK on the other hand has a processing gain which can be 200-300 or more, so the C/N can be greater than the **E_b/η**.

References:

[1]. R.E. Best, “Phase Locked Loops”, Mc Graw Hill, NYC. 1984. (Analyzes the PLL as an FM detector, Includes analysis of the bandwidth efficiency (processing gain) Q and the R factor).

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[6] Bernard Sklar, "Digital Communications", Prentice Hall, 2001.

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